

Acyclicity for Groups and Vector Spaces ^{*†}

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^{*} *Key Words:* Acyclic matching property, Linear matching property, Torsion-free groups.

[†] *AMS Mathematics Subject Classification (2000):* 52B40, 90C27, 20B05

Abstract

A *matching* in an Abelian group G is a bijection f from a subset A to a subset B in G such that $a + f(a) \notin A$, for all $a \in A$. This notion was introduced by Fan and Losonczy who used matchings in \mathbb{Z}^n as a tool for studying an old problem of Wakeford concerning canonical forms for symmetric tensors. The notion of *acyclic matching property* was provided by Losonczy and it was proved that torsion-free groups admit this property. In this paper, we introduce a duality of acyclic matching as a tool for classification of some Abelian groups; moreover, we study matchings for vector spaces and give a connection between matchings in groups and vector spaces. Our tools mix additive number theory, combinatorics and algebra.

1 Introduction

Let G be a group and A and B be two non-empty subsets of G . If $f : A \rightarrow B$ is a matching, we define $m_f : G \rightarrow \mathbb{Z} \cup \{\infty\}$ by $m_f(x) = \#\{a \in A : a + f(a) = x\}$. A matching f is called acyclic if for any matching $g : A \rightarrow B$ with $m_f = m_g$, we have $f = g$. A group G possesses the *finite matching property* if for every pair A and B of non-empty finite subsets satisfying $\#A = \#B$ and $0 \notin B$, there exists at least one matching from A to B . Furthermore, G possesses the *finite acyclic matching property*, if for every pair A and B of non-empty finite subsets satisfying $\#A = \#B$ and $0 \notin B$, there exists at least one acyclic matching from A to B . We say G *fails to have the acyclic matching property at order* $m \in \mathbb{N} \cup \{\infty\}$, if there exist subsets

A and B of G and matchings $f, g : A \rightarrow B$ such that $\#A = \#B = m$, $f \neq g$ and $m_f = m_g$.

Let A be a subset of \mathbb{Z}_p and $f : A \rightarrow A$ be a bijection, where p is prime. Then $\text{ord}_f(a)$ denotes the minimum positive integer n for which $f^n(a) = a$, where $a \in A$. Losonczy in [10] proved the following theorems:

Theorem 1.1. *If G is an Abelian group, then G has the finite matching property if and only if G is torsion-free or cyclic of prime order.*

Theorem 1.2. *If G is an Abelian torsion-free group, then G has the finite acyclic matching property.*

For more results on matchings see [2,4,5,6,7,8,10,11 and 13]. Also, the interested reader is referred to [12] to see more details on Wakeford's problem. Here, we prove the following theorem as a connection between acyclic matching property and its duality.

Theorem 1.3. *Let G be an Abelian group and $G \neq \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$. If G has the finite acyclic matching property, then it fails to have the acyclic matching property at order m , for some $m \in \mathbb{N} \cup \{\infty\}$.*

2 Acyclic matching in a special case for some cyclic groups

In the following theorem, we show that \mathbb{Z}_p fails to have the acyclic matching property at order $\frac{p-1}{2}$ for $p > 5$.

Theorem 2.1. *Let $p > 5$ be a prime. Then \mathbb{Z}_p has the cyclic matching property of order $\frac{p-1}{2}$.*

Proof. Choose a and $b \in \{1, \dots, p-1\}$ such that $a \neq b$, $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) = 1$ and $\left(\frac{a+1}{p}\right) = \left(\frac{b+1}{p}\right) = -1$, where $\left(\frac{\cdot}{p}\right)$ denotes Legendre symbol. See [3] for more results on quadratic residue modulo p . Set $A := \{n^2 : n \in \mathbb{Z}_p \setminus \{0\}\} \subseteq \mathbb{Z}_p$ and define the bijections f and $g : A \rightarrow A$ by $f(n^2) = an^2$ and $g(n^2) = bn^2$ for any $n \in \mathbb{Z}_p \setminus \{0\}$. Now it is clear that f and g are matchings with $m_f = m_g$. This follows \mathbb{Z}_p fails to have the acyclic matching property at order $\frac{p-1}{2}$ for $p > 5$. \square

In the next section, we generalize Theorem 2.1 without invoking the results on quadratic residue.

3 The acyclicity in general case for some cyclic groups

Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ and $f : A \rightarrow A$ be a bijection. If $a \in A$, then $B = \{f^i(a) : i \in \mathbb{N}\}$ is invariant under f , i.e., $f(B) \subseteq B$. It is clear that there exist $a_1, \dots, a_n \in A$ such that $A = \{f^i(a_j) : 1 \leq j \leq n, i \in \mathbb{N}\}$. Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ and $f : A \rightarrow A$ be a matching for which $f^2 \neq id_A$. There exists $a \in A$ with $\text{ord}_f(a) = m > 2$. Now, suppose there exists $b \in A$ such that $b \notin \{f^i(a) : i \in \mathbb{N}\}$ and define $B = \{f^i(b) : i \in \mathbb{N}\}$. Then $f|_{A \setminus B} : A \setminus B \rightarrow A \setminus B$ is a matching with $f \circ f|_{A \setminus B} \neq id_{A \setminus B}$.

In the following theorem, we show that the torsion groups \mathbb{Z}_p fail to have the acyclic matching property at order k , where $2 < k < p-2$. It is a remarkable fact

that $m_f = m_f^{-1}$, where f is a matching from a non-empty subset A of a group G to A and it is applied in the proof of the next theorem. Also, we already have seen in elementary group theory that if the distinct cyclic representation of a permutation $\sigma \in S_n$ has a cycle with a length greater than 2, then $\sigma \neq \sigma^{-1}$.

Theorem 3.1. \mathbb{Z}_p fails to have the acyclic matching property at order k for $2 < k < p - 2$, where p is a prime greater than 5.

Proof. First, we prove that \mathbb{Z}_p fails to have the acyclic matching property at order $p - 3$. Set $A := \mathbb{Z}_p \setminus \{0, 1, p - 1\}$ and define $f : A \rightarrow A$ by

$$f = (4 \ p - 4)(5 \ p - 5) \cdots \left(\frac{p-1}{2} \ \frac{p+1}{2}\right)(3 \ p - 3 \ 2 \ p - 2),$$

where the notation $(a_1 \ a_2 \ \dots \ a_n)$ denotes the permutation of the set $\{a_1, a_2, \dots, a_n\}$ with $a_i \rightarrow a_{i+1}$ $1 \leq i \leq n - 1$ and $a_n \rightarrow a_1$.

Obviously, f is a matching. If $g = f^{-1}$, then $f \neq g$ and $m_f = m_g$. Now, we show that \mathbb{Z}_p fails to have the acyclic matching property at order $p - 4$. Let $A = \mathbb{Z}_p \setminus \{0, 4, p - 4, p - 1\}$. Define $f : A \rightarrow A$ by

$$f = (5 \ p - 5)(6 \ p - 6) \cdots \left(\frac{p-1}{2} \ \frac{p+1}{2}\right)(3 \ p - 3 \ 2 \ p - 2 \ 1).$$

Thus f is a matching. Assume that $g = f^{-1}$, then $f \neq g$ and $m_f = m_g$. This yields G fails to have the acyclic matching property at order $p - 3$ and $p - 4$. If we remove the transpositions of the distinct cyclic representation of f , then f still remains a matching on the omitted subsets and if B_i 's are the omitted subsets, then $f|_{A \setminus B_i} \neq id_{A \setminus B_i}$, for every i , $1 \leq i \leq n$. Suppose $g_i = \left(f|_{A \setminus B_i}\right)^{-1}$, then $f|_{A \setminus B_i} \neq g_i$ and $m_f|_{A \setminus B_i} = m_{g_i}$. Hence \mathbb{Z}_p fails to have the acyclic matching property at orders $p - 3 - 2k$ and $p - 4 - 2k'$ for any $1 \leq k \leq \frac{p-7}{2}$ and $1 \leq k' \leq \frac{p-9}{2}$. Then \mathbb{Z}_p fails to

have the acyclic matching property at order k , for $3 < k < p - 2$. For $k = 3$, define $A = \{1, 2, 4\}$ and $f : A \rightarrow A$ by $f = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix}$. Assuming $g = f^{-1}$ we get the desired result. \square

In the last theorem, we showed that \mathbb{Z}_p fails to have the acyclic matching property at order k , where $2 < k < p - 2$. In the following theorem, we study its behavior at order $p - 2$.

Theorem 3.2. *Let A be a subset of $\mathbb{Z}_p \setminus \{\bar{0}\}$ and $\#A = p - 2$. If $f : A \rightarrow A$ is a matching, then $f^2 = id_A$.*

Proof. Let $f^2 \neq id_A$ and choose $a \in A$ and the positive integer $m > 2$, such that $\text{ord}_f(a) = m$. Thus $f^{i-1}(a) + f^i(a) \notin A$, for each i , $1 \leq i \leq m$. It is clear that $f^{i-1}(a) + f^i(a) \neq f^i(a) + f^{i+1}(a)$ for any i , $1 \leq i \leq m$. Suppose m is even, since $\#A = p - 2$ and $A \cap \{f^{i-1}(a) + f^i(a) : 1 \leq i \leq m\} = \emptyset$, then $f^{i-1}(a) + f^i(a) = f^{i+1}(a) + f^{i+2}(a)$, for any $i = 1, \dots, m - 1$. Let us $a + f(a) = n$ and $f(a) + f^2(a) = n'$, $n = a + f(a) = f^2(a) + f^3(a) = \dots = f^{m-1}(a) + f^m(a) = n$ and $n' = f(a) + f^2(a) = f^3(a) + f^4(a) = \dots = f^{m-2}(a) + f^{m-1}(a) = f^m(a) + a$. Therefore, $\sum_{i=1}^m f^i(a) = (m + 1)n = (m + 1)n'$, so $n = n'$ and it is a contradiction. If m is odd, there exists i , $1 \leq i \leq m$ for which $f^{i-1}(a) + f^i(a) = f^i(a) + f^{i+1}(a)$. Since $\#\{f^{i-1}(a) + f^i(a) : 1 \leq i \leq m\} \leq 2$, therefore $f^2(a) = a$, which is a contradiction. \square

Remark 3.3. There is only one matching f from $\mathbb{Z}_p \setminus \{0\}$ to $\mathbb{Z}_p \setminus \{0\}$. Then \mathbb{Z}_p does not fail to have the acyclic matching property at order $p - 1$.

4 Acyclicity for Abelian torsion-free groups

Theorem 4.1. Let G be an Abelian group. If G is non-divisible and torsion-free, then it fails to have the acyclic matching property at order ∞ .

Proof. Assume that n is the smallest positive integer such that $2nG \subsetneq G$. We break the proof in to the following cases:

Case 1: If $n > 1$, then $G = 2G$. Choose $x \in G \setminus 2nG$ and let $2nG + x = \{2ng + x : g \in G\}$. Define $f, g : 2nG \rightarrow 2nG + x$ by $f(2nt) = 2nt + x$ and $g(2nt) = 2nt + (2n + 1)x$, for any $t \in G$. Since G is torsion-free, f and g are matchings. Choose $g_0 \in G$ such that $x = 2g_0$ and define $A_t = \{y \in G : 4ny + x = t\}$ and $B_t = \{y \in G : 4ny + (2n + 1)x = t\}$, for any $t \in G$. So, $\varphi : A_t \rightarrow B_t$ with $\varphi(y) = y - g_0$ is a bijection. Now, since $m_f(t) = \#A_t$ and $m_g(t) = \#B_t$, then $m_f = m_g$ and G fails to have the acyclic matching property.

Case 2: If $n = 1$, choose $x \in G \setminus 2G$. The bijections $f, g : 2G \rightarrow 2G + x$ defined by $f(2t) = 2t + x$ and $g(2t) = 2t - 3x$ are matchings. Define $A_t = \{y \in G : 4y + x = t\}$ and $B_t = \{y \in G : 4y - 3x = t\}$, for any $t \in G$. Hence $\varphi : A_t \rightarrow B_t$ by $y \mapsto y + x$ is a bijection. This yields that $m_f = m_g$ and G fails to have the acyclic matching property at order ∞ . \square

Example 4.2. For any integer n , $n\mathbb{Z}$ fails to have the acyclic matching property at order ∞ .

In the proof of the Theorem 4.4, the following result on divisible torsion-free groups will be used. See [9] for more details.

Theorem 4.3. Let G be an Abelian group. If G is divisible and torsion-free, then it

is a direct-sum of isomorphic copies of \mathbb{Q} .

By the aforementioned theorem, we can consider \mathbb{Q} as a subset of a group G under the suitable hypotheses on G and we get the following theorem:

Theorem 4.4. Let G be an Abelian group. If G is divisible and torsion-free, then it fails to have the acyclic matching property at order ∞ .

Proof. By Theorem 4.3, \mathbb{Q} is embedded in G . Set $A := \{2k : k \in \mathbb{Z}\}$ and $B := \{2k + 1 : k \in \mathbb{Z}\}$ as subsets of \mathbb{Q} . Define the bijections $f, g : A \rightarrow B$ by $f(2n) = 2n + 1$ and $g(2n) = 2n + 5$. It is clear that f and g are matchings. Now, if $x \in G \setminus \{4k + 1 : k \in \mathbb{Z}\}$, then $m_f(x) = m_g(x) = 0$. On the other hand, if $x \in \{4k + 1 : k \in \mathbb{Z}\}$, then $m_f(x) = m_g(x) = 1$ and then, in all cases $m_f = m_g$. \square

Corollary 4.5. By Theorem 4.1 and Theorem 4.4, if G is an Abelian, torsion-free group then G fails to have the acyclic matching property at order ∞ .

Example 4.6. Two additive groups \mathbb{R} and \mathbb{Q} fails to have the acyclic matching property at order ∞ .

Now, our result regarding the connection of matching properties for Abelian groups.

Theorem 4.7. Suppose G is an Abelian group and $G \neq \mathbb{Z}_2, \mathbb{Z}_3$ and \mathbb{Z}_5 . If G has the finite acyclic matching property, then it fails to have the acyclic matching property at order m for some $m \in \mathbb{N} \cup \{\infty\}$.

We will see the proof of this theorem in section 7.

5 Linear version of acyclicity for subspaces in a field extension

Let G be an Abelian group and f and $g : A \rightarrow B$ be two matchings where A and B are non-empty finite subsets of G and $m_f = m_g$. For any $x \in G$, define $A_x^f = \{a \in A : a + f(a) = x\}$, $A_x^g = \{a \in A : a + g(a) = x\}$, $\mathcal{A}_f = \{A_x^f : m_f(x) \neq 0\}$ and $\mathcal{A}_g = \{A_x^g : m_g(x) \neq 0\}$. It is clear that \mathcal{A}_f and \mathcal{A}_g are distinct decompositions for A and $\#\mathcal{A}_f < \infty$, $\#\mathcal{A}_g < \infty$. Define the function $\varphi : A \rightarrow A$ by the following method:

Define $\mathcal{A}_f = \{A_{x_1}^f, A_{x_2}^f, \dots, A_{x_m}^f\}$. Since $m_f = m_g$, then $\mathcal{A}_g = \{A_{x_1}^g, A_{x_2}^g, \dots, A_{x_m}^g\}$. Assume that $a_1 \in A_{x_1}^f$, choose an arbitrary element b_1 of $A_{x_1}^g$ and put $\varphi(a_1) = b_1$. If a_2 is another element of $A_{x_1}^f$, choose another arbitrary element b_2 of $A_{x_1}^g \setminus \{b_1\}$ and put $\varphi(a_2) = b_2$. We can continue this procedure to define φ on $A_{x_1}^f$ and by the similar way, we can define the function φ on whole A which is bijective and satisfies $a + f(a) = \varphi(a) + g(\varphi(a))$ for any $a \in A$.

Conversely, assume that f and g are two matchings from A to B and there exists a bijection $\varphi : A \rightarrow A$ for which $a + f(a) = \varphi(a) + g(\varphi(a))$, for any $a \in A$. We claim that $m_f = m_g$. Let us x be an arbitrary element of G . We have the following cases:

Case 1: If $x \in A$, according to the definition of matching, $m_f(x) = m_g(x) = 0$.

Case 2: If $x \notin A$, then $m_f(x) = \#\{a \in A : a + f(a) = x\} = \#\{a \in A : \varphi(a) + g(\varphi(a)) = x\} = \#\{a \in A : a + g(a) = x\} = m_g(x)$. So, $m_f = m_g$, as claimed.

So we get the following theorem:

Theorem 5.1. Let A , B , f and g be as above. Then, $m_f = m_g$ if and only if there

exists a bijection $\varphi : A \rightarrow A$ such that $a + f(a) = \varphi(a) + g(\varphi(a))$, for any $a \in A$.

By the aforementioned theorem, a natural generalization for the acyclic matching in vector spaces is inspired. To see this concept, we need to present some definitions from [5].

Definition 5.2. Let $K \subseteq L$ be a field extension and A, B be n -dimensional K -subspaces of the field extension L . Let $\mathcal{A} = \{a_1, \dots, a_n\}$ and $\mathcal{B} = \{b_1, \dots, b_n\}$ be bases of A and B , respectively. It is said \mathcal{A} is *matched* to \mathcal{B} if

$$a_i b \in A \Rightarrow b \in \langle b_1, \dots, \hat{b}_i, \dots, b_n \rangle$$

for all $b \in B$ and $i = 1, \dots, n$, where $\langle b_1, \dots, \hat{b}_i, \dots, b_n \rangle$ is the hyperplane of B spanned by the set $\mathcal{B} \setminus \{b_i\}$; moreover, it is stated that A is *matched* with B if every basis \mathcal{A} of A can be matched to a basis \mathcal{B} of B . It is said that L has the *linear matching property* if, for every $n \geq 1$ and every n -dimensional subspaces A, B of L with $1 \notin B$, the subspace A is matched with B . A *strong matching* from A to B is a linear isomorphism $\varphi : A \rightarrow B$ such that any basis \mathcal{A} of A is matched to the basis $\varphi(\mathcal{A})$ of B .

Now, we are in the situation to give the linear version of acyclicity.

Definition 5.3. Let $K \subseteq L$ be a field extension and A and B be two n -dimensional K -subspaces in L such that $n > 1$. Let $f, g : A \rightarrow B$ be two strong matchings. We say f is equivalent to g and denote it by $f \sim g$ if there exists a linear isomorphism $\varphi : A \rightarrow A$ such that $af(a) = \varphi(a)g(\varphi(a))$, for any $a \in A$; moreover, we state that the strong matching $f : A \rightarrow B$ is *linear acyclic matching* if for any strong matching $g : A \rightarrow B$, if $f \sim g$, then $f = cg$, for some $c \in K$. We say $K \subseteq L$ *fails to have the linear acyclic matching property* at order $m \in \mathbb{N}$, if there exist K -subspaces A

and B in L and strong matchings f and $g : A \rightarrow B$ such that $f \neq g$, $f \sim g$ and $\dim_K A = \dim_K B = m$.

Eliahou and Lecouvey in [5] proved the following theorems. The interested reader is also referred to [1].

Theorem 5.4. Let $K \subset L$ be a field extension. Then K has the linear matching property if and only if L contains no proper finite-dimensional extension over K .

Theorem 5.5. Let $K \subset L$ be a field extension and A and B be n -dimensional K -subspaces distinct from $\{0\}$. There is a strong matching from A to B if and only if $AB \cap A = \{0\}$. In this case, any isomorphism $\varphi : A \rightarrow B$ is a strong matching.

Now, our result regarding the connection of the linear matching properties for field extensions.

Theorem 5.6. Let $K \subsetneq L$ be a field extension admit the linear matching property and $\#K \geq 5$. Then it fails to have the linear acyclic matching property at order m , for some $m \in \mathbb{N}$.

We will see the proof of this theorem in section 7.

6 The linear acyclicity of a given order

In this section, we study the linear acyclicity for finite field extensions.

Theorem 6.1. Let $K \subsetneq L$ be a field extension with $[L : K] = n$, $\#K \geq 5$ and no

proper intermediate subfield. Then $K \subset L$ fails to have the linear acyclic matching property at order m , for any $1 \leq m \leq (n+1)/4$.

Proof. Choose $m \in \mathbb{N}$ and $a \in L \setminus K$ for which $m \leq (n+1)/4$. Set $A_m := \langle a, a^3, \dots, a^{2m-1} \rangle$. Then $A_m \cap A_m^2 = \{0\}$, because $K(a) = L$, for any $A \in L \setminus K$. Using Theorem 5.5, there exists a strong matching $f_m : A_m \rightarrow A_m$. Next, set $g_m := f_m^{-1}$. One more time using Theorem 5.5, follows that f_m^{-1} is a strong matching. Now, if $f_m \circ f_m \neq id_{A_m}$, then $f_m \neq g_m$ and $f_m \sim g_m$. On the other hand, if $f_m \circ f_m = id_{A_m}$, choose $c \in K$ such that $c^2 \notin \{0, 1\}$. Set $h_m := c^{-2}g_m$, then h_m is a strong matching. We claim $f_m \sim h_m$. In order to prove, define $\varphi_m := cf_m$. We get $af_m(a) = \varphi_m(a)h_m(\varphi_m(a))$, for any $a \in A_m$. This tells us $f_m \sim h_m$, as claimed. \square

Theorem 6.2. Let $K \subset L$ be a purely transcendental extension. Then, it fails to have the linear acyclic matching property at order m , for any $m \in \mathbb{N}$.

Proof. Let a be an element of $L \setminus \{0, 1\}$ and set $A_m := \langle a, a^3, \dots, a^{2m-1} \rangle$. Then $A_m \cap A_m^2 = \{0\}$ and by Theorem 5.5, there exists a strong matching f_m from A_m to A_m . By the same method in the previous theorem we can conclude that $K \subset L$ fails to have the acyclic linear matching property at order m , for any $m \in \mathbb{N}$. \square

Remark 6.3. If a field extension $K \subseteq L$ has no finite-dimensional proper intermediate field extension and $\#K \geq 5$. Then, it fails to have the acyclic matching property at order m , for some $m \in \mathbb{N}$.

Proof. This is a direct consequence of Theorems 6.1 and 6.2. \square

7 Main results

Theorem 7.1. Suppose G is an Abelian group and $G \neq \mathbb{Z}_2, \mathbb{Z}_3$ and \mathbb{Z}_5 . If G has the finite acyclic matching property, then it fails to have the acyclic matching property at order m for some $m \in \mathbb{N} \cup \{\infty\}$.

Proof. Assume G has the finite acyclic matching property. Then G has the finite matching property. Using Theorem 1.1, G is cyclic of prime order or torsion-free. Invoking Corollary 4.5 and Theorem 2.1, G fails to have the acyclic matching property at order m for some $m \in \mathbb{N} \cup \{\infty\}$. \square

Theorem 7.2. Let $K \subsetneq L$ be a field extension admit the linear matching property and $\#K \geq 5$. Then it fails to have the linear acyclic matching property at order m , for some $m \in \mathbb{N}$.

Proof. If $K \subset L$ has the linear matching property, so Theorem 5.4 yields it has no proper finite-dimensional K -subspaces and by Remark 6.3, $K \subset L$ fails to have the acyclic matching property at order m , for some $m \in \mathbb{N}$. \square

Acknowledgement: The authors are thankful to Professors Noga Alon and Saieed Akbari for providing useful comments and discussions.

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